Europ. J. Combinatorics (1999) **20**, 759–771
Article No. eujc.1999.0342
Available online at http://www.idealibrary.com on IDEAL

Generalized Quadrangles with a Spread of Symmetry

BART DE BRUYN[†]

We present a common construction for some known infinite classes of generalized quadrangles. Whether this construction yields other (unknown) generalized quadrangles is an open problem. The class of generalized quadrangles obtained this way is characterized in two different ways. On the one hand, they are exactly the generalized quadrangles having a spread of symmetry. On the other hand, they can be characterized in terms of the group of projectivities with respect to a spread. We explore some properties of these generalized quadrangles. All these results can be applied to the theory of the glued near hexagons, a class of near hexagons introduced by the author in De Bruyn (1998) *On near hexagons and spreads of generalized quadrangles*, preprint.

1. INTRODUCTION AND OVERVIEW

A *generalized quadrangle* of order (s, t) is an incidence system of points and lines satisfying the following conditions.

(1) Every two different points are incident with at most one line, or equivalently every two different lines meet in at most one point.
(2) Every point is incident with exactly $t + 1$ lines ($t \geq 1$) and every line is incident with exactly $s + 1$ points ($s \geq 1$).
(3) For every line L and every point p not incident with L, there exists a unique point q and a unique line M such that $p \mathrm{~I~} M \mathrm{~I~} q \mathrm{~I~} L$ (here I denotes the incidence relation).

The point q in (3) is called the *projection* of p on L. If $p \mathrm{~I~} L$, then the projection of p on L is p itself. Generalized quadrangles (GQs for short) were introduced in [13]. We refer to [7] and [12] for the most important results about GQs. A GQ of order (s, t) is sometimes denoted by GQ(s, t). The point-line dual of a GQ(s, t) is a GQ(t, s). *Grids* (respectively *dual grids*) are exactly the GQs of order $(s, 1)$ (respectively $(1, t)$). A GQ is called *trivial* when it is a grid or a dual grid. An *ovoid* (respectively *spread*) of a GQ is a set of points (respectively lines) such that every line (respectively point) of the GQ is incident with exactly one element of the set. In Section 3 we will describe a common construction for some known infinite families of GQs (see Section 8 for examples). In general, the GQs arising from this construction have a lot of spreads. Whether the construction yields other (unknown) GQs is an interesting but open problem. The construction makes use of what is called here an admissible triple; this triple consists of a Steiner system, a group and a map that satisfy certain properties. The GQs which are obtained this way are characterized in the following sections. We prove that the subgroup of the automorphism group of a GQ(s, t), $s \neq 1$, fixing a spread of the generalized quadrangle contains at most $s + 1$ elements, with equality (S is called then a spread of symmetry) if and only if the GQ can be derived from an admissible triple. We also prove that a certain group, called the group of projectivities with respect to a spread (of a GQ(s, t), $s \neq 1$), contains at least $s + 1$ elements, with equality if and only if the GQ is derivable from an admissible triple. In Sections 6 and 7, we will determine all spreads of symmetry of some classes of generalized quadrangles. In Section 7, we also prove that an association scheme can be associated to a GQ(s, s^2), $s \neq 1$, having a spread of symmetry. In [4], a new construction for near hexagons

[†]Research Assistant of The Fund of Scientific Research – Flanders (Belgium).

0195–6698/99/080759 + 13 $30.00/0

(to be defined later) is presented. The near hexagons arising this way are called glued. The construction makes use of two GQs with a spread in each of them. The last section proves that both spreads are spreads of symmetry and as a consequence all results from the sections before are important for the theory of the glued near hexagons.

2. The Construction of Payne

If x, y are two noncollinear points of a GQ(s, t), then the *hyperbolic line* through x and y is the set of all points collinear with all $t+1$ common neighbours of x and y. A pair $\{x, y\}$ of noncollinear points of a GQ(s, t) is called *regular* if the hyperbolic line through them contains exactly $t+1$ elements (i.e., the maximal number of such elements). A point x is called *regular* when $\{x, y\}$ is regular for all points y not collinear with x. Dually, the notion of regularity can be defined for lines of a GQ.

If x is a regular point of a GQ $\mathcal{Q}$ of order (s, s), then a GQ $P(\mathcal{Q}, x)$ of order $(s-1, s+1)$ can be constructed as follows (see [6] and [7]). The points of $P(\mathcal{Q}, x)$ are the points of $\mathcal{Q}$ not collinear with x; the lines of $P(\mathcal{Q}, x)$ are the lines of $\mathcal{Q}$ not incident with x together with all the hyperbolic lines through x; incidence is the natural one.

Examples. The above construction can be applied to the GQ $W(q)$, whose points are the points of PG$(3, q)$ and whose lines are the totally isotropic lines of PG$(3, q)$ with respect to a symplectic polarity ζ. Every point x of $W(q)$ is regular and we find the following description for $P(W(q), x)$. The points are the points of PG$(3, q)$ not in the plane x^ζ; the lines are the lines of $W(q)$ not in x^ζ together with those lines of PG$(3, q)$ through x but not contained x^ζ; incidence is the natural one. If q is even then $P(\mathcal{Q}, x)$ is isomorphic to $T_2^*(O)$ with O a regular hyperoval (i.e., a conic union its nucleus), if q is odd then $P(\mathcal{Q}, x)$ is isomorphic to $AS(q)$. The above construction can also be applied to the GQ $T_2(O)$ with O an oval of PG$(2, 2^h)$, $h \geq 1$. This GQ has at least one regular point and several regular lines, see [7].

3. A Construction of Generalized Quadrangles

Definition. An incidence system of points and blocks is called a Steiner system with parameters (τ, k, v) if: (i) there are exactly v points; (ii) every block is incident with exactly $k > 1$ points; (iii) every τ different points are incident with exactly one block. Such a Steiner system is often denoted by $S(\tau, v, k)$. A *partial spread* of a Steiner system is a set of mutually disjoint blocks.

We describe now a procedure which will give generalized quadrangles.

1. Start with a Steiner system $\mathcal{D} = S(2, k, v)$ with $1 < k \leq v$ and let $\mathcal{P}$ be the set of its points. The number of blocks through a point is $\frac{v-1}{k-1}$. Hence, we can put $k = s+1$ and $v = st + 1$ with s and t strict positive integers. The total number of blocks is equal to $\frac{v(v-1)}{k(k-1)}$, hence $(s+1)|t(t-1)$.
2. Find a group G of order $s+1$ and a map $\Delta : \mathcal{P} \times \mathcal{P} \mapsto G, (x, y) \mapsto \delta_{xy}$ such that the points x, y, z are collinear if and only if $\delta_{xy}\delta_{yz} = \delta_{xz}$ (multiplicative notation for the group G). If this holds, then we say that the triple $(\mathcal{D}, G, \Delta)$ is *admissible*. In this case, one has that $\delta_{xx} = e$ and $\delta_{yx} = \delta_{xy}^{-1}$ for all $x, y \in \mathcal{P}$ (in the following e always denotes the identity element of the group G). Hence $\delta_{xy}\delta_{yz}\delta_{zx} = e$ for collinear points $x, y, z \in \mathcal{P}$.

The following theorem states that every admissible triple yields a generalized quadrangle.

THEOREM 3.1. *Suppose that* $(\mathcal{D}, G, \Delta)$ *is an admissible triple. Let* Γ *be the graph with vertex set* $G \times \mathcal{P}$*; two different vertices* (g_1, x) *and* (g_2, y) *are adjacent whenever:*

(a) $x = y$ *and* $g_1 \neq g_2$*, or*
(b) $x \neq y$ *and* $g_2 = g_1\delta_{xy}$.

Then Γ *is the collinearity graph of a* GQ(s, t).

PROOF. The graph Γ contains $(1 + s)(1 + st)$ vertices and every vertex is adjacent to $s(t + 1)$ others. We will prove that every two adjacent vertices are contained in a unique maximal clique and that this clique contains exactly $s + 1$ elements. If we consider these cliques as the lines of a geometry $\mathcal{Q}$ with the vertices of Γ as points, then every point of $\mathcal{Q}$ is incident with $t + 1$ lines and the number of points at distance at most one to a fixed line is $(s+1)+(s+1)ts = (s+1)(1+st)$. Hence $\mathcal{Q}$ is a GQ(s, t).

Now, suppose that $p_1 = (g_1, x)$ and $p_2 = (g_2, y)$ are two adjacent vertices of Γ; we determine how the common neighbours (g_3, z) look. If $x = y \neq z$, then $g_3 = g_1\delta_{xz} = g_2\delta_{xz}$, implying that $g_1 = g_2$, a contradiction. Hence if $x = y$, then p_1 and p_2 are in a unique maximal clique containing all the points (g, x) with $g \in G$. If $x \neq y$, then also $x \neq z \neq y$ and $g_3 = g_1\delta_{xz} = g_2\delta_{yz} = g_1\delta_{xy}\delta_{yz}$. This implies that $\delta_{xz} = \delta_{xy}\delta_{yz}$ or that $z \in xy$. It now easily follows that p_1 and p_2 are contained in a unique maximal clique, namely $\{(g_1\delta_{xz}, z)|z \mathrm{I}\, xy\}$. □

If there is an admissible triple with $\mathcal{D}$ and G as components, then there are a lot of admissible triples with $\mathcal{D}$ and G as components, as we will show now in the following two properties; however all the corresponding GQs turn out to be isomorphic.

PROPERTY 1. Let $(\mathcal{D}, G, \Delta)$ be an admissible triple. With every point x of $\mathcal{D}$, we associate an element $\delta_x \in G$. Put $\Delta'(x, y) = \delta'_{xy} = \delta_x^{-1}\delta_{xy}\delta_y$ for every two points x, y of $\mathcal{D}$. Then:

(1) $(\mathcal{D}, G, \Delta')$ is an admissible triple;
(2) the two corresponding GQs are isomorphic.

PROOF. Part (1) is easily verified. Now, let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the generalized quadrangles corresponding to $(\mathcal{D}, G, \Delta)$ and $(\mathcal{D}, G, \Delta')$, respectively. The map $(g, x) \mapsto (g\delta_x, x)$ defines then an isomorphism between $\mathcal{Q}_1$ and $\mathcal{Q}_2$. □

PROPERTY 2. Let $(\mathcal{D}, G, \Delta)$ be an admissible triple and let θ be an automorphism of the group G. Put $\Delta''(x, y) = \delta''_{xy} = \delta_{xy}^{\theta}$ for every two points x, y of $\mathcal{D}$. Then:

(1) $(\mathcal{D}, G, \Delta'')$ is an admissible triple;
(2) the two corresponding GQs are isomorphic.

PROOF. Part (1) is easily verified. Now, let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the generalized quadrangles corresponding to $(\mathcal{D}, G, \Delta)$ and $(\mathcal{D}, G, \Delta'')$ respectively. The map $(g, x) \mapsto (g^{\theta}, x)$ defines then an isomorphism between $\mathcal{Q}_1$ and $\mathcal{Q}_2$. □

THEOREM 3.2. *Let* $\mathcal{Q}$ *be a generalized quadrangle arising from an admissible triple* $(\mathcal{D}, G, \Delta)$. *For each* $x \in \mathcal{P}$*, let* $L_x = \{(g, x)|g \in G\}$. *Then* $S = \{L_x|x \in \mathcal{P}\}$ *is a spread of* $\mathcal{Q}$ *and there exist* $s + 1$ *different automorphisms of* $\mathcal{Q}$ *fixing each line of* S.

PROOF. The first statement is clear. For each $h \in G$, the map $\theta_h : \mathcal{P} \mapsto \mathcal{P}, (g, x) \mapsto (hg, x)$ defines clearly an automorphism that fixes each line of S. □

THEOREM 3.3. *With each partial spread S_1 of $\mathcal{D}$, there corresponds a spread S_1' of $\mathcal{Q}$.*

PROOF. Let $S_1 = \{L_1, \ldots, L_r\}$ be a partial spread of $\mathcal{D}$ $(0 \leq r \leq \frac{1+st}{1+s})$. For each $i \in \{1, \ldots, r\}$, let x_i be a given point of L_i. If $L_{(g,x_i)}$ denotes the line $\{(g\delta_{x_i y_i}, y_i) | y_i \mathrm{~I~} L_i\}$ for each $g \in G$ and $i \in \{1, \ldots, r\}$, then $S_1' = \{L_{(g,x_i)} | g \in G; 1 \leq i \leq r\} \cup \{L_x | x$ is not incident with $L_i, \forall i \in \{1, \ldots, r\}\}$ is a spread of $\mathcal{Q}$. □

Suppose that $(\mathcal{D}, G, \Delta)$ is an admissible triple. If $\mathcal{D}$ is not a line $(t > 1)$, then the fact that there are at least as many blocks as points implies that $t \geq s + 1$. If $t = s + 1$, then $\mathcal{D}$ is a $S(2, s+1, s^2+s+1)$ and hence a projective plane, but we will prove that this is impossible. Hence $s = 1$ or $s + 2 \leq t \leq s^2$ (every nontrivial GQ(s, t) satisfies $t \leq s^2$, see [5] and [7]).

THEOREM 3.4. *There are no admissible triples where the Steiner system is a projective plane.*

PROOF. If there was such an admissible triple, then there would exist a GQ of order $(s, s+1)$ with $s \geq 2$, but this contradicts the condition $(s + t) \mid st(s + 1)(t + 1)$ holding for a GQ(s, t), see [7]. □

Every known GQ has one of the following parameters:

(I) $(s, 1)$ or $(1, t)$ with $s, t \geq 1$;
(II) $(q - 1, q + 1)$ or $(q + 1, q - 1)$ with q a prime power;
(III) (q, q) with q a prime power;
(IV) (q^2, q^3) or (q^3, q^2) with q a prime power;
(V) (q, q^2) or (q^2, q) with q a prime power.

We will prove (see Section 8), that all grids and dual grids (case (I)) are derivable from an admissible triple. If $t = s + 2$ then $\mathcal{D}$ is an $S(2, s + 1, (s + 1)^2)$ and hence an affine plane. Note that $(s, t) = (q^2, q^3)$ cannot occur since the above condition $(s + 1) \mid t(t - 1)$ is not satisfied. Finally, if $t = s^2$ then $\mathcal{D}$ is an $S(2, s + 1, s^3 + 1)$.

4. AUTOMORPHISMS FIXING A SPREAD OF A GENERALIZED QUADRANGLE

Let $S = \{L_1, \ldots, L_{1+st}\}$ be a spread of a generalized quadrangle $\mathcal{Q}$ of order (s, t) and let G_S be the group of automorphisms of $\mathcal{Q}$ which fix each line of S.

THEOREM 4.1. *If $\mathcal{Q}$ is not a grid, then each nontrivial element of G_S has no fix points. As a consequence $|G_S| = \frac{s+1}{n}$ with n some nonzero integer.*

PROOF. Suppose that x is a fixpoint of $\theta \in G_S$.

(a) Suppose $y \sim x$ with $xy \notin S$. If $L \in S$ is incident with y, then $x \mathrm{~I~} xy \mathrm{~I~} y \mathrm{~I~} L$ and $x \mathrm{~I~} xy^\theta \mathrm{~I~} y^\theta \mathrm{~I~} L$ implies that $y^\theta = y$.
(b) Suppose $y \not\sim x$, then x and y have a common neighbour z such that $xz, yz \notin S$. From (a) it follows that $z^\theta = z$ and $y^\theta = y$.
(c) Suppose that $y \sim x$ with $xy \in S$. Take a point z such that $x \not\sim z \sim y$. Then (b) implies that $z^\theta = z$ and (a) implies that $y^\theta = y$.

The automorphism θ hence is trivial. The group G_S acts as a permutation group on the set of points of any line of S. If there are n orbits on such a line, then $|G_S| = \frac{s+1}{n}$. □

If $\mathcal{Q}$ is an $(s+1) \times (s+1)$-grid, then there are exactly $(s+1)!$ elements in G_S and the above theorem does not hold when $s > 1$.

DEFINITION. A *spread of symmetry* is a spread S satisfying the following property: for every $K, L \in S$ and every two lines M and N meeting K and L, there exists an automorphism $\theta \in G_S$ such that $M^\theta = N$. If $\mathcal{Q}$ is a grid, then the two spreads are spreads of symmetry. If $\mathcal{Q}$ is not a grid, then S is a spread of symmetry if and only if $|G_S| = s+1$.

The previous theorem states that $|G_S| \leq s+1$ in the case that $\mathcal{Q}$ is not a grid. This is very interesting in view of Theorem 3.2. Also the converse of this theorem is true.

THEOREM 4.2. *The generalized quadrangle $\mathcal{Q}$ is derivable from an admissible triple if and only if $|G_S| \geq s+1$ for some spread S of $\mathcal{Q}$. If $\mathcal{Q}$ is not a grid, then the condition $|G_S| \geq s+1$ is equivalent to $|G_S| = s+1$.*

PROOF. We will prove that $\mathcal{Q}$ is derivable from an admissable triple if there exists a spread S for which $|G_S| \geq s+1$. Theorems 3.2 and 4.1 then complete the proof. Every grid can be constructed from an admissible triple (see Section 8.1). Hence, we may suppose that $\mathcal{Q}$ is not a grid. In this case G_S contains exactly $s+1$ elements.

1. Construction of the Steiner system $S(2, s+1, 1+st)$.
   The points of the Steiner system are the lines of S. The blocks are obtained as follows. Take a line $T \notin S$ of $\mathcal{Q}$. Let $R_1, \ldots, R_{s+1}$ be the lines of S intersecting T; then $\{R_1, \ldots, R_{s+1}\}$ is a line of the Steiner system. It remains to show that every two lines R_1 and R_2 of S are contained in a unique block. Every line intersecting R_1 and R_2 will yield such a block. If T is a line intersecting R_1 and R_2, then $A = \{T^h | h \in G_S\}$ is a set of $s+1$ lines intersecting R_1 and R_2, hence these are all the lines intersecting R_1 and R_2. If $L \in S$ intersects T in a point t, then L intersects T^h in t^h. Hence, every element of A will yield the same block through R_1 and R_2.
2. Construction of the group G and the map Δ.
   We take $G = G_S$. Let $L_1 \in S$ and let $p \in L_1$ be fixed. Take two points $x = L_i$ and $y = L_j$ of the Steiner system. Let x_1 be the projection of p on the line L_i (in the generalized quadrangle $\mathcal{Q}$), let x_2 be the projection of x_1 on the line L_j and finally let x_3 be the projection of x_2 on the line L_1. Now, there exists an element $\theta \in G_S$ such that $x_3 = p^\theta$ and we put $\Delta(x, y) = \delta_{xy} := \theta^{-1}$. It remains to show that the points $x = L_i, y = L_j, z = L_k$ of the Steiner system are collinear if and only if $\delta_{xy}\delta_{yz} = \delta_{xz}$. Put $\delta_{xy} = \alpha^{-1}$, $\delta_{yz} = \beta^{-1}$ and $\delta_{xz} = \gamma^{-1}$. Denote by p_l ($l \in \{1, i, j, k\}$) the projection on the line L_l in the generalized quadrangle $\mathcal{Q}$. Put $a = p_i(p), b = p_j(a), c = p_k(b), d = p_k(a)$, then $p^\gamma = p_1(d)$. From $p \sim p_j(p) \sim p_k p_j(p) \sim p^\beta$ and $p^\alpha \sim b \sim c \sim p_1(c)$, it follows then that $p^{\beta\alpha} = p_1(c)$. Now,
   $$\begin{aligned} \delta_{xy}\delta_{yz} = \delta_{xz} &\Leftrightarrow \gamma = \beta\alpha \\ &\Leftrightarrow c = d \\ &\Leftrightarrow a, b, c \text{ are on a line } T \\ &\Leftrightarrow x, y, z \text{ are collinear.} \end{aligned}$$
3. Isomorphism between $\mathcal{Q}$ and the GQ from $(\mathcal{D}, G, \Delta)$.
   Let z be an arbitrary point of $\mathcal{Q}$. Let L_z denote the unique line of S incident with z and let z' denote the projection of z on the fixed line L_1. There exists a unique $g_z \in G_S$ such that $z' = p^{g_z}$. We prove now that the map $z \mapsto (g_z^{-1}, L_z)$ defines an isomorphism

between the GQs. It is clearly a bijection and since both geometries have the same parameters, it suffices to show that adjacency is preserved. So, let z_1 and z_2 be two adjacent points of $\mathcal{Q}$ and put $x = L_{z_1}$ and $y = L_{z_2}$. From $p^{g_{z_1}} \sim z_1 \sim z_2 \sim p^{g_{z_2}}$, it follows that $g_{z_2} = \delta_{xy}^{-1} g_{z_1}$ or $g_{z_2}^{-1} = g_{z_1}^{-1}\delta_{xy}$. Hence $(g_{z_2}^{-1}, y) \sim (g_{z_1}^{-1}, x)$. □

DEFINITION. A spread S is called *normal* when each pair $\{L, M\} \subseteq S$ is regular and when all the members of the hyperbolic line through $\{L, M\}$ also belong to S. From item 1 in the proof of the previous theorem, it follows that each spread of symmetry is a normal spread.

Consider now the following problem. Given a nontrivial GQ $\mathcal{Q}$, determine all spreads of symmetry. In the next theorem, we will show how these spreads can be determined, but first we define the following linear space $\mathcal{L}(\mathcal{Q})$: the points of $\mathcal{L}(\mathcal{Q})$ are the lines of $\mathcal{Q}$; the lines of $\mathcal{L}(\mathcal{Q})$ are the pencils (the sets of lines through a point of $\mathcal{Q}$) together with the hyperbolic lines (on the set of lines of $\mathcal{Q}$); incidence is the natural one.

THEOREM 4.3. *If S is a spread of symmetry of a nontrivial GQ $\mathcal{Q}$, then $S = \langle K, L, M\rangle$ for any three non-$\mathcal{L}(\mathcal{Q})$-collinear points $K, L, M \in S$.*

PROOF. Let $K, L, M \in S$ be three non-$\mathcal{L}(\mathcal{Q})$-collinear points; then $\langle K, L, M\rangle \subseteq S$ contains at least $s(s+1)+1$ elements. If $N \in S$ do not lie in $\langle K, L, M\rangle$, then $\langle K, L, M, N\rangle \subseteq S$ contains at least $s(s^2+s+1)+1$ elements. Hence, $1+st \geq s^3+s^2+s+1$ or $t \geq s^2+s+1$, a contradiction, since $t \leq s^2$ holds for an arbitrary nontrivial GQ(s, t). □

If $|G_S| > 1$, then the condition $(s+t) \mid st(s+1)(t+1)$ can be strengthened.

LEMMA 4.4 ([1]). *If f is the number of fixpoints of an automorphism θ and if g is the number of points x for which $x^\theta \neq x \sim x^\theta$, then $(t+1)f + g \equiv 1 + st \pmod{s+t}$.*

COROLLARY 4.5. *If $|G_S| > 1$, then $(s+t) \mid s(s+1)(t+1)$.*

PROOF. Let θ be any nontrivial element of G_S and apply the previous lemma. □

5. CHARACTERIZATIONS IN TERMS OF THE GROUP OF PROJECTIVITIES

DEFINITION. Let L and M be two disjoint lines of a generalized quadrangle $\mathcal{Q}$. The projection on M determines a bijection, denoted by $[L, M]$, from the set of points of L to the set of points of M. The bijection $[L, M]$ is called the *perspectivity* from L to M. If $L_1, \dots, L_k$ are $k \geq 1$ lines of $\mathcal{Q}$ such that L_i and L_{i+1} $(1 \leq i \leq k-1)$ are disjoint, then the composition $[L_1, L_2] \dots [L_{k-1}, L_k]$ is called a *projectivity from L_1 to L_k*. If $L_k = L_1$, then we find a permutation on the point set of L_1. The set of all such permutations on the point set of L_1 is a group, denoted by $\Pi(L_1)$, and it is called *the group of projectivities of* L_1. If L_1 is a line of a spread S, then in the above definition of projectivity, one can require that all lines L_i belong to the spread. In this way, one can define the *group of projectivities of L_1 with respect to S*, denoted by $\Pi_S(L_1)$.

Now, let $S = \{L_1, \dots, L_{1+st}\}$ be a spread of a generalized quadrangle $\mathcal{Q}$ of order (s, t). Put

$$\delta_{ij} = [L_1, L_i][L_i, L_j][L_j, L_1],$$
$$A = \{\delta_{ij} | 1 \leq i, j \leq 1+st\}.$$

It is clear that A generates $\Pi_S(L_1)$. If $\mathcal{Q}$ is a grid, then $\Pi(L_1)$ and $\Pi_S(L_1)$ are trivial groups; if $\mathcal{Q}$ is not a grid, then the following theorem states something about the number of elements of $\Pi_S(L_1)$.

THEOREM 5.1. *Suppose that $\mathcal{Q}$ is not a grid. If x, y are points of L_1, then there exists an element $a \in A$, such that $x^a = y$. Hence $|A|, |\Pi_S(L_1)| \geq s+1$.*

PROOF. Let $L \neq L_1$ be a fixed line of $\mathcal{Q}$ through y, let L_i be any line of S disjoint with L, let x' be the projection of x on L_i, let x'' be the projection of x' on L and let L_j be the line of S through x''. Now $\delta_{ij} \in A$ maps x to y. □

For the rest of this section, we are interested in the case of equality. We will prove that equality occurs if and only if $\mathcal{Q}$ is not a grid and S is a spread of symmetry of $\mathcal{Q}$.

LEMMA 5.2. *Let X be a set of order $n \geq 1$. If G is a regular group of permutations on X, then there are precisely $s+1$ permutations of X which commute with every element of G. These $s+1$ permutations form a regular group $\tilde{G}$ of permutations on X.*

PROOF. Let $x \in X$ be fixed. If h commutes with every element of G, then $(x^g)^h = (x^h)^g$ for all $g \in G$, hence the image of x under h determines completely h. Now, for every element $y \in X$, one can define the permutation h_y as follows:

$$(x^g)^{h_y} = y^g, \qquad \forall g \in G.$$

It is straightforward to check that h_y commutes with every element of G and that h_y is the trivial permutation if it has at least one fixpoint. □

THEOREM 5.3. *If $\theta \in G_S$ then θ induces a permutation $\bar{\theta}$ on the point set of L_1 that commutes with each element of $\Pi_S(L_1)$. Conversely, if a permutation ϕ on the point set of L_1 commutes with each element of $\Pi_S(L_1)$, then $\phi = \bar{\theta}$ for some $\theta \in G_S$.*

PROOF. First, we notice that $\phi = \bar{\theta}$ determines θ completely. For, let $x \in L_i$ be an arbitrary point of $\mathcal{Q}$. From $d(x, x^{[L_i,L_1]}) \leq 1$, it follows that $d(x^\theta, x^{[L_i,L_1]\phi}) \leq 1$ and hence

$$x^\theta = x^{[L_i,L_1]\phi[L_1,L_i]}. \tag{1}$$

Next, let ϕ be a permutation on the point set of L_1 and let us determine under which conditions the map θ defined by (1) is an automorphism. If $x \in L_i$ and $y \in L_j$ (with $i \neq j$) are two collinear points, then $y = x^{[L_i,L_j]}$ and

$$\begin{aligned} x^\theta \sim y^\theta &\Leftrightarrow x^{[L_i,L_1]\phi[L_1,L_i]} \sim y^{[L_j,L_1]\phi[L_1,L_j]} \\ &\Leftrightarrow x^{[L_i,L_1]\phi[L_1,L_i][L_i,L_j]} = x^{[L_i,L_j][L_j,L_1]\phi[L_1,L_j]}. \end{aligned}$$

This holds for every $x \in L_i$ if and only if

$$\begin{gathered} \phi[L_1, L_i][L_i, L_j][L_j, L_1] = [L_1, L_i][L_i, L_j][L_j, L_1]\phi, \\ \phi\delta_{ij} = \delta_{ij}\phi. \end{gathered}$$

This holds for all i, j if and only if ϕ commutes with each element of $\Pi_S(L_1)$. □

THEOREM 5.4. *If $|\Pi_S(L_1)| = s+1$ then also $|G_S| = s+1$. Conversely, if $|G_S| = s+1$ and if $\mathcal{Q}$ is not a (2×2)-grid, then $|\Pi_S(L_1)| = s+1$.*

PROOF. If $|\Pi_S(L_1)| = s+1$, then $\mathcal{Q}$ is not a grid and Theorem 5.1 implies that $\Pi_S(L_1)$ is a regular group of permutations of L_1. Lemma 5.2 then implies that there are exactly $s+1$ permutations of L_1 which commute with each element of $\Pi_S(L_1)$ and by Theorem 5.3 each of these permutations can be extended to an element of G_S. Hence $|G_S| = s+1$. Conversely, suppose that $\mathcal{Q}$ is not a grid and that $|G_S| = s+1$; then Theorem 4.1 implies that G_S induces a regular group of permutations on the point set of L_1. Lemma 5.2 and Theorem 5.3 then imply that $|\Pi_S(L_1)| \leq s+1$ and the result follows by Theorem 5.1. □

We prove now the following theorem.

THEOREM 5.5. *If $\Pi_S(L_1)$ is commutative then S is a spread of symmetry and $\mathcal{Q}$ is derivable from an admissible triple.*

PROOF. We may suppose that $\mathcal{Q}$ is not a grid. Theorems 5.1 and 5.3 imply that $|G_S| \geq |\Pi_S(L_1)| \geq s+1$, hence $|G_S| = s+1$ by Theorem 4.1. So S is a spread of symmetry. □

6. THE CASE $t = s + 2$

The case where S is a spread of symmetry of a GQ $\mathcal{Q}$ of order $(s, s+2)$ has already been considered in [11].

DEFINITION. Let p be a point of a nontrivial GQ(s, t). A *symmetry about* p is an automorphism of GQ(s, t) fixing p and every point collinear with p. One can prove that there are at most t such symmetries; if there are exactly t symmetries, then p is called a *centre of symmetry*. A centre of symmetry is always a regular point.

REMARK. If x is a regular point of a GQ $\mathcal{Q}$ of order (s, s), then the s^2 hyperbolic lines through x define a spread of $\mathcal{P}(\mathcal{Q}, x)$.

We mention now the following result proved in [11].

THEOREM 6.1 ([11]). *A GQ $\mathcal{Q}$ of order $(s, s+2)$ has a spread of symmetry if and only if $\mathcal{Q} \simeq P(\mathcal{Q}', (\infty))$ with $\mathcal{Q}'$ a* GQ$(s+1, s+1)$ *and (∞) a centre of symmetry of $\mathcal{Q}'$. The spread of symmetry corresponds with the hyperbolic lines through (∞).*

Let $(\mathcal{D}, G, \Delta)$ be an AT where $\mathcal{D}$ is an affine plane of order $s+1$, then by the previous theorem there exists a GQ of order $(s+1, s+1)$ having a centre of symmetry. This result was already in [3].

We determine now all spreads of symmetry of all known GQs of order $(s, s+2)$.

$T_2^*(O)$

THEOREM 6.2. *Consider the GQ $T_2^*(O)$ with O a hyperoval of the plane* PG$(2, q)$ *which is embedded as hyperplane in* PG$(3, q)$. *For each $h \in O$, the set S_h of all lines of* PG$(3, q)$ *through h and not contained in* PG$(2, q)$ *is a spread of symmetry of $T_2^*(O)$. Conversely, if $q \neq 2$ every spread of symmetry of $T_2^*(O)$ arises this way.*

PROOF. The q translations of AG$(3, q)$ determined by the point h at infinity yield q automorphisms of $T_2^*(O)$ fixing each line of S_h. Conversely, suppose that S is a spread of symmetry; then every line-pair of S is regular. Since $q \neq 2$, these two lines determine the same point at infinity, see Theorem 3.3.4 of [7]. Hence, the result follows. □

$AS(q)$

THEOREM 6.3. *Let $\mathcal{Q}$ be the GQ $AS(q)$ with $q \neq 3$ and consider the model $P(W(q), x)$ of $AS(q)$. The hyperbolic lines of $W(q)$ through x define then the unique spread of symmetry of $AS(q)$.*

PROOF. If S is a spread of symmetry, then every line-pair of S is regular. Since $q \neq 3$, these two lines are hyperbolic lines of $W(q)$ through x, see Theorem 3.3.5 of [7]. The set of all hyperbolic lines of $W(q)$ through x is a spread of symmetry by Theorem 6.1. □

REMARK. If $q = 3$, then the above theorem does not hold. However, $AS(3) \simeq Q(5, 2)$ and in Section 7.1 we will determine all spreads of symmetry of $Q(5, q)$.

$\underline{(S^-_{xy})^D}$

Let O be a hyperoval of $\Pi_\infty = \mathrm{PG}(2, 2^h)$, $h \geq 2$, and let x, y be two points of O. The GQ $T_2(O \setminus \{x\})$ has a lot of regular lines, see [7], and by Section 2 generalized quadrangles of order $(2^h + 1, 2^h - 1)$ can be constructed. However, we use here the model S^-_{xy} given in [8]. Embed Π_∞ as a hyperplane in Π. The points are of three types:

(1) points of $\Pi \setminus \Pi_\infty$;
(2) planes through x not containing y;
(3) planes through y not containing x.

The lines of S^-_{xy} are the lines of Π not contained in Π_∞ and intersecting O in a point different from x and y. Incidence is the natural one, i.e., a point and a line of S^-_{xy} are incident when they are incident as objects of Π. The points of type (2), respectively (3), form an ovoid O_{xy}, respectively O_{yx}, of S^-_{xy}. These ovoids are normal, see [10]. If $(S^-_{xy})^D$, the dual generalized quadrangle of S^-_{xy}, is isomorphic to $T^*_2(O)$, then there are q other normal ovoids, see [9] when this situation precisely occurs. If $(S^-_{xy})^D$ is not isomorphic to $T^*_2(O)$, then O_{xy} and O_{yx} are the only normal ovoids of S^-_{xy}, see [10]. The ovoids O_{xy} and O_{yx} are both 'ovoids of symmetry'. The q automorphisms of Π fixing each point of Π_∞ and every line through x, respectively y, induce the q automorphisms of S^-_{xy} fixing each point of O_{xy}, respectively O_{yx}.

7. THE CASE $t = s^2$

7.1. The GQ $Q(5, q)$. We determine all spreads of symmetry of $Q(5, q)$ and we reason in the dual GQ $H(3, q^2)$. Let O be the ovoid of $H(3, q^2)$ corresponding to the spread of symmetry S. We will prove that O is the intersection of $H(3, q^2)$ with a nontangent plane (such an intersection consists of $q^3 + 1$ mutually noncollinear points, hence it is an ovoid). Each pair $\{x, y\}$ of noncollinear points of $H(3, q^2)$ is regular and the hyperbolic line through them consists of the $q + 1$ points in the intersection of the line xy with the Hermitian variety. Let $x, y, z \in O$, where z is not a member of the hyperbolic line through $\{x, y\}$. From Theorem 4.3, it follows that O is contained in the intersection of $H(3, q^2)$ with the plane α through x, y, z. Hence, α is nontangent and $O = \alpha \cap H(3, q^2)$. Let ζ be the unitary polarity of $\mathrm{PG}(3, q^2)$ corresponding to $H(3, q^2)$ and put $\langle \bar{u} \rangle = \alpha^\zeta$. The unitary reflections with centre $\langle \bar{u} \rangle$ and axis π generate a subgroup of order $q + 1$ fixing each point of O pointwise. Hence O is an ovoid of symmetry.

7.2. The existence of an association scheme

DEFINITION. An *association scheme* is a pair $(X, \mathcal{R})$ with X a finite nonempty set and $\mathcal{R} = \{R_0, \ldots, R_n\}$ a partition of $X \times X$ such that the following conditions are satisfied.

(1) $R_0 = \{(x, x) | x \in X\}$.

(2) $R_i^T \in \{R_0, \dots, R_n\}$ for all $i \in \{0, \dots, n\}$ ($R_i^T = \{(y, x) | (x, y) \in R_i\}$).
(3) There exist integers p_{jk}^i, called the *parameters of the association scheme*, having the following property: for all $(x, y) \in R_i$, there exist exactly p_{jk}^i elements $z \in X$ such that $(x, z) \in R_j$ and $(z, y) \in R_k$.

If $R_i^T = R_i$, for all $i \in \{0, \dots, n\}$, then the association scheme is called *symmetric*.

Now, suppose that $\mathcal{Q}$ is a nontrivial GQ(s, s^2) having a spread of symmetry S and let L be a fixed line of S. Put $G = \Pi_S(L)$. Let $X = S \setminus \{L\}$ and $\mathcal{R} = \{R_0\} \cup \{R_g | g \in G\}$; $(M, N) \in R_0$ if and only if $M = N$, $(M, N) \in R_g$ if and only if $M \neq N$ and $[L, M][M, N][N, L] = g$. In the following theorem, we will prove that $(X, \mathcal{R})$ is an association scheme. We have that $p_{0\beta}^\alpha = p_{\beta 0}^\alpha = 1$ or 0 depending on $\alpha = \beta$ or $\alpha \neq \beta$. For $g, h \in G$, we have that $p_{gh}^0 = n_g$ or 0 depending on $gh = e$ or $gh \neq e$. The other parameters together with the parameters n_g are given in the following theorem.

THEOREM 7.1. *$(X, \mathcal{R})$ is an association scheme and the remaining parameters are $(g, h, k$ denote elements of G, different from $e)$:*

$$
\begin{array}{lll@{\qquad}lll}
n_e & = & s-1; & n_g & = & s^2-1; \\
p_{ee}^e & = & s-2; & p_{ee}^g & = & 0; \\
p_{ge}^e & = & 0; & p_{eg}^e & = & 0; \\
p_{ge}^g & = & 0; & p_{eg}^g & = & 0; \\
p_{gh}^e & = & 0 \text{if} gh = e; & p_{gh}^e & = & s+1 \textit{ if } gh \neq e; \\
p_{eh}^g & = & 1 \textit{ if } h \neq g; & p_{he}^g & = & 1 \textit{ if } h \neq g; \\
p_{hk}^g & = & s+1 \textit{ if } g \neq hk; & p_{hk}^g & = & 1 \textit{ if } g = hk.
\end{array}
$$

PROOF. We leave the proof to the reader. It follows from the fact that every three mutually noncollinear points of a GQ(s, s^2) have exactly $s+1$ common neighbours, see [2] and [7]. □

The above-defined association scheme is symmetric if and only if $g^{-1} = g$ for all $g \in G$. From $ab = a^{-1}b^{-1} = (ba)^{-1} = ba, \forall a, b \in G$, it follows that G is an elementary abelian group and hence equal to the additive group of some finite field with characteristic 2.

8. EXAMPLES

In the previous sections, we determined all spreads of symmetry for some generalized quadrangles. Theorem 4.2 can then be applied to determine the corresponding admissible triple.

8.1. *The trivial GQs*

(1) Let $\mathcal{D}$ be a line of length $s+1$ $(t = 1)$ and let G be any group of order $s+1$. For all $x, y \in \mathcal{P}$ we put $\delta_{xy} = e$. The corresponding generalized quadrangle is an $(s+1) \times (s+1)$-grid.
(2) Let $\mathcal{D}$ be an $S(2, 2, t+1)$ (i.e., the complete graph K_{t+1}), let $G = \{e, a\}$ be the group of order 2. Put $\delta_{xy} = e$ if $x = y$ and $\delta_{xy} = a$ if $x \neq y$. The corresponding generalized quadrangle is a dual grid.

8.2. The GQ $P(W(q), x)$. Let $\mathcal{D}$ be the Desarguesian affine plane AG$(2, q)$ with point set $\mathcal{P} = \{(x_1, x_2) | x_1, x_2 \in \mathrm{GF}(q)\}$, let G be the additive group of GF(q) and put $\delta_{xy} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$ for two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $\mathcal{D}$. If q is odd, then $\mathcal{Q} \simeq AS(q)$; if q is even, then $\mathcal{Q} \simeq T_2^*(O)$ with O a regular hyperoval.

8.3. The GQ $T_2^*(O)$. Let $O = \{o, o_1, \ldots, o_{q+1}\}$ be a hyperoval of $\mathrm{PG}(2, q)$ and let S be the spread of $T_2^*(O)$ determined by the point $o \in O$. Let L_∞ be a line of $\mathrm{PG}(2, 2^h)$ disjoint with O and let $\mathcal{D}$ be the associated affine plane. After coordinatizing $\mathcal{D}$, we have $o \leftrightarrow (k, l)$ and $o_i \leftrightarrow (k_i, l_i)$ for all $i \in \{1, \ldots, q+1\}$. Let G be the additive group of $\mathrm{GF}(q)$. We define now $\Delta(x, y)$ for every two points $x \leftrightarrow (x_1, x_2)$ and $y \leftrightarrow (y_1, y_2)$. If $x = y$, then we define $\Delta(x, y) := 0$; if $x \neq y$, then there exist a unique $i \in \{1, \ldots, q+1\}$ and a unique λ such that $(x_1 - y_1, x_2 - y_2) = \lambda(k_i - k, l_i - l)$ and we define $\Delta(x, y) := \lambda$. The corresponding GQ is isomorphic to $T_2^*(O)$.

8.4. The GQ $Q(5, q)$. Let the vector space $V(3, q^2)$ be equipped with a nonsingular Hermitian form $(\cdot,\cdot)$ (i.e., $(\sum \mu_i v_i, \sum \lambda_j w_j) = \sum\sum \mu_i \lambda_j^q (v_i, w_j)$) and let U be the corresponding unital of $\mathrm{PG}(2, q^2)$. There is a Steiner system $\mathcal{D} = S(2, q+1, q^3+1)$ related to U (the blocks are the intersections of U with nontangent lines). Let $G = \{\alpha \in \mathrm{GF}(q^2) | \alpha^{q+1} = 1\}$ with multiplication inherited from $\mathrm{GF}(q^2)$. Let $z = \langle \bar{a} \rangle$ be a given point of U. For two points $x = \langle \bar{b} \rangle$ and $y = \langle \bar{c} \rangle$ of U, we define

$$\begin{aligned}\Delta(x, y) &= -(\bar{a}, \bar{b})^{q-1}(\bar{b}, \bar{c})^{q-1}(\bar{c}, \bar{a})^{q-1} \in G \qquad \text{if } x, y, z \text{ are mutually different;}\\ &= 1 \qquad \text{otherwise.}\end{aligned}$$

$\Delta(x, y)$ is well defined. If we replace $\bar{b}$ by $\mu\bar{b}$ and $\bar{c}$ by $\lambda\bar{c}$ with $\mu, \lambda \in \mathrm{GF}(q^2) \setminus \{0\}$, then the above value for $\Delta(x, y)$ is unaltered. The corresponding GQ is isomorphic to $Q(5, q)$.

8.5. The GQ $(S_{xy}^-)^D$ Let O be a hyperoval of $\mathrm{PG}(2, 2^h)$, $h \geq 2$, and let $x, y \in O$. Let S be the spread of $(S_{xy}^-)^D$ corresponding to the ovoid O_{xy} of S_{xy}^-. We can coordinatize Π_∞ in such a way that $x \leftrightarrow (1, 0, 0)$, $y \leftrightarrow (0, 1, 0)$ and such that the point with coordinates $(0, 0, 1)$ belongs to O. We then can write

$$O = \{(1, 0, 0), (0, 1, 0)\} \cap \{(f(\lambda), \lambda, 1) | \lambda \in \mathrm{GF}(q)\},$$

where $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ is a function satisfying

(i) f is a bijection;
(ii) $\frac{f(x_2)-f(x_1)}{x_2-x_1} \neq \frac{f(x_3)-f(x_1)}{x_3-x_1}$ for every three distinct elements x_1, x_2, x_3 of $\mathrm{GF}(2^h)$;
(iii) $f(0) = 0$.

Now, let $\mathcal{D}$ be the affine plane $\mathrm{AG}(2, 2^h)$ and let G be the additive group of $\mathrm{GF}(q)$. Now, take two points $p_1 = (\alpha_1, \beta_1)$ and $p_2 = (\alpha_2, \beta_2)$ of $\mathrm{AG}(2, 2^h)$. We put $\Delta(p_1, p_2)$ equal to $\frac{[f(\alpha_1)-f(\alpha_2)][\beta_1-\beta_2]}{\alpha_1-\alpha_2}$ if $\alpha_1 \neq \alpha_2$ and 0 otherwise. The corresponding GQ is isomorphic to $(S_{xy}^-)^D$.

PROBLEM. Besides the examples give above, are there other GQs derivable from an admissible triple?

9. APPLICATIONS TO NEAR HEXAGONS

A *near hexagon* is an incidence system of points and lines such that the following conditions are satisfied.

(1) Every two different points are incident with at most one line.

(2) For every point p and every line L, there is a unique point q on L, nearest to p (distances are measured in the collinearity graph).
(3) The maximal distance between two points is 3.

In [4], the author presented the following construction for near hexagons. Let $\mathcal{Q}_i$, $i \in \{1, 2\}$, be a GQ(s, t_i), let $S_i = \{L_1^{(i)}, L_2^{(i)}, \ldots, L_{1+st_i}^{(i)}\}$ be a spread of $\mathcal{Q}_i$ and let θ be a bijection from $L_1^{(1)}$ to $L_1^{(2)}$ (the lines of the GQs are regarded as sets of points). Consider then the following graph Γ on the vertex set $L_1^{(1)} \times S_1 \times S_2$. Two different vertices $(x, L_i^{(1)}, L_j^{(2)})$ and $(y, L_k^{(1)}, L_l^{(2)})$ are adjacent whenever at least one of the following conditions is satisfied:

(1) $i = k$ and $x^{\theta[L_1^{(2)}, L_j^{(2)}]} \sim y^{\theta[L_1^{(2)}, L_l^{(2)}]}$ (in $\mathcal{Q}_2$),
(2) $j = l$ and $x^{[L_1^{(1)}, L_i^{(1)}]} \sim y^{[L_1^{(1)}, L_k^{(1)}]}$ (in $\mathcal{Q}_1$).

It is proved in [4] that any two adjacent vertices are contained in a unique maximal clique. If we regard these maximal cliques as the lines of a geometry with point set equal to the vertex set of Γ, then this geometry is a near hexagon if and only if

$$[\Pi_{S_1}(L_1^{(1)}), \theta\Pi_{S_2}(L_1^{(2)})\theta^{-1}] = 0.$$

Here 0 denotes the trivial group and $[\cdots, \cdots]$ is the group generated by all commutators $[g, \theta h\theta^{-1}]$ with $g \in \Pi_{S_1}(L_1^{(1)})$ and $h \in \Pi_{S_2}(L_1^{(2)})$. A near hexagon constructed this way is called *glued*. Suppose now that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are not grids; then $|\theta\Pi_{S_2}(L_1^{(2)})\theta^{-1}| \geq s+1$ and by Theorem 5.3, we know that S_1 is a spread of symmetry. Similarly, S_2 is a spread of symmetry of Q_2. Hence, the results of the previous sections have implications in the theory of the glued near hexagons. In particular, we have the following results.

(1) Not all GQs are good for the above construction: the conditions $t \geq s + 2$ if $s, t \neq 1$, $(s + 1) \mid t_i(t_i - 1)$ and $(s + t_i) \mid s(s + 1)(t_i + 1)$ exclude some classes of (known) GQs (see Section 3 for a survey).
(2) For several classes of GQs, we determined all spreads of symmetry (see Sections 6 and 7). There exists an example of a glued near hexagon for every such spread, see [4].

The automorphisms of $\mathcal{Q}_i$ fixing each line of S_i $(i \in \{1, 2\})$ can be recovered as follows in the near hexagon. For $j \in \{1, \ldots, 1 + st_2\}$, let $A_j = \{(x, L_i^{(1)}, L_j^{(2)}) | x \in L_1^{(1)}, 1 \leq i \leq 1 + st_1\}$. The lines of the near hexagon entirely contained in A_j induce a GQ(s, t_1), denoted by $\mathcal{Q}'_j$. For every point x of the near hexagon there exists a unique point $x' \in A_j$ nearest to x, see [4]. As above, we can define projections $[A_i, A_j]$ for all $i, j \in \{1, \ldots, 1 + st_2\}$ and these projections preserve adjacency. It is straightforward to check that the map $[A_1, A_i][A_i, A_j][A_j, A_1]$, $1 \leq i, j \leq 1 + st_2$, defines an automorphism of $\mathcal{Q}'_1$ fixing each line of the spread $S = \{M_1, \ldots, M_{1+st_1}\}$ with $M_i = \{(x, L_i^{(1)}, L_1^{(2)}) | x \in L_1^{(1)}\}$.

Acknowledgements

The fact that the GQ $(S_{xy}^{-})^D$ has a spread of symmetry was not in a first version of this paper. It was after a discussion with S. E. Payne that this example was studied and finally included in this paper. M. Brown communicated to me that a result mentioned here (see the remark following Theorem 6.1) was already proven by him. He proved in [3] that GQs can be constructed from admissible triples where the design is an affine plane.

REFERENCES

1. C. T. Benson, On the structure of generalized quadrangles, *J. Algebra*, **15** (1970), 443–454.
2. R. C. Bose and S. S. Shrikhande, Geometric and pseudo-geometric graphs $(q^2 + 1, q + 1, 1)$, *J. Geom.*, **2** (1972), 75–94.
3. M. R. Brown, Generalized quadrangles and associated structures, Ph. D. Thesis, 1997.
4. B. De Bruyn, On near hexagons and spreads of generalized quadrangles, preprint, 1998.
5. D. G. Higman, Partial geometries, generalized quadrangles and strongly regular graphs, in: *Atti Convegno di Geometria e sue Applicazioni*, A. Barlotti (ed.), Perugia, 1971, pp. 263–293.
6. S. E. Payne, Nonisomorphic generalized quadrangles, *J. Algebra*, **18** (1971), 201–212.
7. S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, *Research Notes in Mathematics*, **110**, Pitman, Boston, 1984.
8. S. E. Payne, Hyperovals and generalized quadrangles, in C. A. Baker and L. A. Batten (eds), *Finite Geometries Lecture Notes in Pure and Applied Mathematics*, **103**, M. Dekker, 1985, pp. 251–271.
9. S. E. Payne, Hyperovals yield many GQ, *Simon Stevin*, **60** (1986), 211–225.
10. S. E. Payne and M. A. Miller, Collineations and characterizations of generalized quadrangles of order $(q - 1, q + 1)$, *Suppl. Rend. Circolo Mat. Palermo*, **53** (1998), 137–166.
11. M. De Soete and J. A. Thas, A coordinatization of generalized quadrangles of order $(s, s + 2)$, *J. Comb. Theory*, **48** (1988), 1–11.
12. J. A. Thas, Generalized polygons, in: *Handbook of Incidence Geometry, Buildings and Foundations*, chapter 9, F. Buekenhout (ed.), North-Holland, Amsterdam, 1995, pp. 383–432.
13. J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Publ. Math. IHES*, **2** (1959), 14–60.

Received 18 January 1999 and accepted in revised form 21 July 1999

BART DE BRUYN
Department of Pure Mathematics and Computer Algebra,
University of Ghent,
Galglaan 2,
B-9000 Gent,
Belgium
E-mail: bdb@cage.rug.ac.be